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Numerical and analytical investigations of localisation in magnetic fields

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Abstract. Using a standard tight-binding model, the dependence of the localisation length ξ on a perpendicular magnetic field in quasi-one-dimensional systems is investigated. A well known numerical method is used to calculate the localisation length as a function of the number of flux quanta per unit cell α and other system parameters. An attempt to explain the $\xi(\alpha)$ curves perturbatively yields qualitative agreement and corrects the earlier results for ξ as a function of energy and disorder W in the limit of large W obtained by similar techniques. Finally, the Lloyd model is re-examined with a magnetic field included. Previous claims of an exact solution for the Lloyd model for $B = 0$ have been attacked but, we believe, not rigorously defeated. We hope to rehabilitate the Lloyd model by demonstrating its abilities in the magnetic field.

1. Introduction

A common approach to the localisation problem in two or three dimensions is based on finite size scaling and strip geometry [1–3]. Apart from a few details, there is general agreement about what can be expected from this method and how to judge the data obtained so far. But with a magnetic field included, there has been no systematic investigation of the whole parameter space (made up of the energy, the disorder, the magnetic field, the topological ϑ parameter—see section 2—and the system size), which would justify a concluding statement about universality (cf [4]) for the unitary case. Even for the orthogonal case, this question is still open [5]. There have been, however, examinations of scaling and the mobility edge problem for a fixed magnetic field, like those of Schweitzer *et al* [6] or Ando [7].

We took this as a motivation to examine systematically the field dependence of localisation lengths in quantum Hall systems. This was done numerically by applying a method introduced by Pichard [1, 2] which will be briefly reviewed in the following section. For our present purposes, the numerical results are used merely to check the analytical ones. It is possible, however, to go further and calculate the critical exponents of the localisation length from the data and investigate the above-mentioned hypothesis of universality in the localisation problem which has been questioned recently by MacKinnon [5]. Results on that will be published elsewhere.

Two attempts to get a glimpse of real understanding of what is seen numerically (a perturbative and a non-perturbative one) based on the previous analytic $B = 0$ work of Johnston and Kunz [8, 9] follow in sections 3 and 4. Their results are generalised to

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include a non-zero magnetic field and an error which makes their perturbative results disagree with the numerical ones is removed from their calculation.

2. Numerical results

Our model Hamiltonian

$$H = \sum_{i=1}^{N}\sum_{x=1}^{l}(\varepsilon_x^i \mid ix\rangle\langle ix \mid + \mid i+1\,x\rangle\langle ix \mid + \mid i-1\,x\rangle\langle ix \mid + \exp[2\pi\mathrm{i}(\alpha i+\vartheta)] \mid i\,x+1\rangle\langle ix \mid + \exp[-2\pi\mathrm{i}(\alpha i+\vartheta)] \mid i\,x-1\rangle\langle ix \mid) \tag{1}$$

incorporates the magnetic field via periodic Peierls factors (ϑ labelling non-equivalent represention of the translation group in a 2D system with simple periodic boundary conditions) and diagonal disorder via matrix elements ε_x^i taken from a box distribution of width W. Indices $i, j, k, \ldots$, run along the strip axis ($1, \ldots, N$, $N \to \infty$); indices $x, y, z, \ldots$, denote the l sites in each cross section. As usual, we employ the transfer matrix representation

$$\boldsymbol{u}_N = \mathbf{T}(N)\boldsymbol{u}_0 \tag{2}$$

of Schrödinger's equation for the 2D strip with

$$\begin{aligned} &\boldsymbol{u}_n = (\psi(n+1,1), \ldots, \psi(n+1,l), \psi(n,1), \ldots, \psi(n,l)) \\ &\mathbf{T}(N) = \prod_{i=1}^{N}\mathbf{T}_i \qquad \boldsymbol{u}_{n+1} = \mathbf{T}_n\boldsymbol{u}_n \qquad \mathbf{T}_n = \begin{pmatrix} \mathbf{P}_n & -\mathbf{1}_l \\ \mathbf{1}_l & \mathbf{0} \end{pmatrix} \end{aligned} \tag{3}$$

and

$$\mathbf{P}_n = \begin{pmatrix} E-\varepsilon_1^n & -(J(\alpha))^{-1} & 0 & \ldots & 0 & -(J(\alpha))^{-1} \\ -J(\alpha) & E-\varepsilon_2^n & -(J(\alpha))^{-1} & \ldots & 0 & 0 \\ 0 & -J(\alpha) & E-\varepsilon_3^n & \ldots & 0 & 0 \\ & \ddots & \ddots & \ddots & & \\ 0 & 0 & \ldots & & E-\varepsilon_{l-1}^n & -(J(\alpha))^{-1} \\ -(J(\alpha))^{-1} & 0 & \ldots & & -(J(\alpha))^{-1} & E-\varepsilon_1^n \end{pmatrix} \tag{4}$$

with $J(\alpha) = \exp[2\pi\mathrm{i}(\alpha n+\vartheta)]$ and $(J(\alpha))^{-1} = \exp[-2\pi\mathrm{i}(\alpha n+\vartheta)]$, to define the localisation length $\xi_l(E, W, B)$ by the Lyapunov exponent of the smallest modulus γ_1

$$\xi_l = \gamma_1^{-1} \tag{5}$$

of the limiting matrix

$$\boldsymbol{\Gamma} = \lim_{N\to\infty}(\mathbf{T}(N)\mathbf{T}^{\dagger}(N))^{1/2N} \tag{6}$$

the existence of which is guaranteed by Oseledec's famous theorem [10].

An alternative approach frequently used by Kramer *et al* (e.g. [6]) is to define the localisation length by the off-diagonal Green function $(\langle 1x \mid G(z) \mid Ny\rangle)_{xy}$ which can be calculated recursively. This yields the density of states as a useful by-product, but only γ_1 from all the Lyapunov exponents. To check a certain result of the analytic treatment of the Lloyd model, however, we need the sum of all exponents which we obtain by a method first introduced by Pichard [1, 2]: Oseledec's theorem enables us to calculate γ_k from

$$\lim_{N\to\infty} \frac{1}{N} \ln\left(\frac{\|\mathbf{T}(N)\boldsymbol{u}\|}{\|\boldsymbol{u}\|}\right) = \gamma_k \qquad \forall\, \boldsymbol{u} \in V_k \tag{7}$$

if the starting vector u is taken from the eigenspace V_k of $\exp(\gamma_k)$. However, this cannot directly be used to calculate γ_k numerically, because we require N to be about 2×10^5 to keep the statistical error

$$\frac{\Delta_n \gamma_1}{\gamma_1} = \left[\frac{1}{n-1}\left(\frac{n\sum_{k=1}^{n}(\ln b_k^{(1)})^2}{(\sum_{k=1}^{n}\ln b_k^{(1)})^2} - 1\right)\right]^{1/2} \tag{8}$$

below 1%. That is, matrix elements of $\mathbf{T}(N)$ will become as large as about $(W/J)^{(2\times 10^5)}$ where J is the nearest-neighbour hopping matrix element. In order to avoid overflow errors occurring during the calculation, we change coordinates after about every fifth iteration by using

$$\boldsymbol{B}'_k = \left(\boldsymbol{B}_k - \sum_{j<k}(\boldsymbol{B}'_j, \boldsymbol{B}_k)\boldsymbol{B}'_j\right) b^{(k)^{-1}} \tag{9}$$

where

$$\mathbf{T}(N) = \begin{pmatrix} | & | & & | \\ \boldsymbol{B}_1 & \boldsymbol{B}_2 & \cdots & \boldsymbol{B}_l \\ | & | & & | \end{pmatrix} \tag{10}$$

and

$$b^{(k)} = \left\| \boldsymbol{B}_k - \sum_{j<k}(\boldsymbol{B}'_j, \boldsymbol{B}_k)\boldsymbol{B}'_j \right\|. \tag{11}$$

In other words: we orthonormalise the columns of $\mathbf{T}(k)$, so that the first column develops freely yielding the largest Lyapunov exponent, the second grows everywhere but in the direction of V_l thus yielding the next largest exponent etc. Therefore the columns of $\mathbf{T}(N)$ will point into the eigendirections of $\mathcal{T}$ after the iteration has converged.

Furthermore, as we are allowed to calculate

$$\gamma = \lim_{N\to\infty} \frac{1}{N} \ln\left(\frac{\|\mathbf{T}(N)u\|}{\|\mathbf{T}^{(N-1)}(N-1)u\|} \cdots \frac{\|\mathbf{T}''(2)u\|}{\|\mathbf{T}'(1)u\|} \frac{\|\mathbf{T}'(1)u\|}{\|u\|}\right) \tag{12}$$

instead of (7), denoting the product of n transfer matrices after m transformations of the type (9) by $\mathbf{T}^{(m)}(n)$, we obtain the γ_k after expanding the telescope fraction in (12) by accumulating the relative growth of each column $\boldsymbol{B}_k$:

$$\gamma_k = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \ln b_n^{(k)}. \tag{13}$$

Let us first look at the raw data for $\xi_l(\alpha)$. Fixing $l = 3$ (this is comparatively small but taking this value we save considerable computer time since $\xi(\alpha)$ looks *qualitatively* the same for all l—as can be checked by a few points for large systems) and varying W as a parameter, we get figure 1. The numerical method becomes unreliable for $\alpha < 10^{-5}$ (the hopping matrix elements of (1) will not be sampled properly unless $\alpha N > 1$) or for αs of order unity (the discretised model breaks down when the magnetic length $\ell = \alpha^{-1/2}$ starts to approach the discretisation length). So what we see is a localisation length which remains constant over some three orders of magnitude of α, starting at $\alpha = 10^{-6}$, and increases quadratically for very high fields ($\alpha \approx 10^{-2}$, see figure 4).

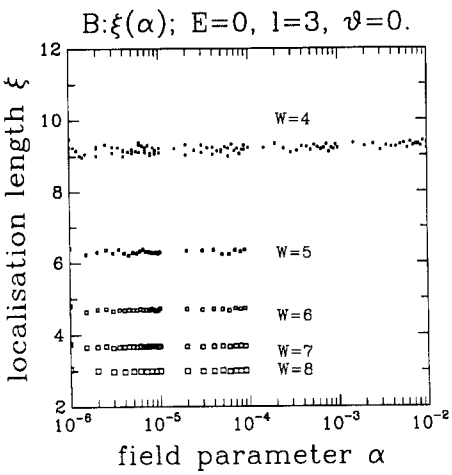

Figure 1. Numerical values for the localisation length as a function of the magnetic field, $\xi(\alpha)$, with W as a parameter, for small systems ($l = 3$).

An attempt to understand these features analytically is put forward in the next section.

3. Perturbation theory

A first attempt to understand the $\xi(\alpha)$ curves analytically relies on an idea of Johnston and Kunz [8] who define the localisation length by

$$\xi_l(E, W, B) = -\left(\lim_{N\to\infty} \frac{1}{N} \Re\left\{\left\langle \lim_{\varepsilon\to+0} \ln G^N_{xx}(E + \mathrm{i}\varepsilon)\right\rangle\right\}\right)^{-1} \tag{14}$$

and rewrite (1) as

$$H = W(H_0 + \lambda H_1) = W\left(\sum_{ix} \tau_x^i \mid ix\rangle\langle ix \mid + \lambda H_1\right) \tag{15}$$

$$\equiv W\tilde{H} \qquad \tau_x^i \in \tfrac{1}{2}[-1, 1]$$

to be able to use a locator expansion in powers of W^{-1} for the off-diagonal Green function to evaluate (14). Unfortunately, their result is not capable of predicting the

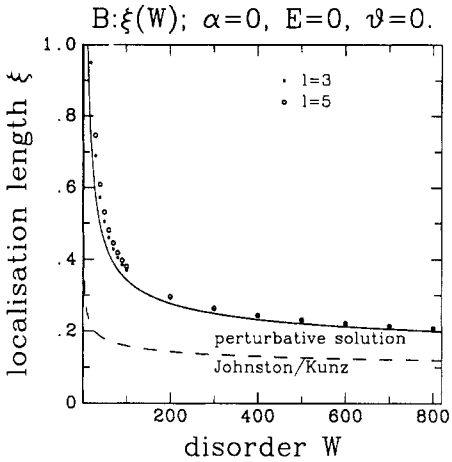

Figure 2. $\xi(W)$ for large disorder W. The circles are the numerical values for $l = 3$ and $l = 5$, the full curve denotes the perturbative solution and the broken one the old result of Johnston and Kunz [8].

numerical values even in the limit of very large disorder (see figure 2). Here we repeat their calculation and correct it for d dimensions and a non-vanishing magnetic field.

We start from a locator series of the form

$$G_{ix,ky} = \frac{1}{W}\sum_{n=0}^{\infty} \sum_{\substack{\gamma^{(n)} \\ (i,x)\gamma(k,y)}} L_x^i \prod_{j=1}^{n} \lambda J_{\gamma^{(n)}(j-1),\gamma^{(n)}(j)} L_{\gamma^{(n)}(j)}. \tag{16}$$

for a matrix element $\langle ix \mid G(z) \mid ky\rangle$ of G, where we have to sum over all paths $\gamma^{(n)}$ of length n, connecting the sites (i,x) and (k,y) and consisting of a hopping matrix element $\lambda J_{i'x',k'y'}$ for each link and a locator $L_{x'}^{i'} = [(z/W) - \tau_{x'}^{i'}]^{-1}$ for each site along the path.

For a first try, we keep only the first two non-vanishing orders of (16) which can be visualised as in figure 3: they are paths of length $N-1$ for the first, and of length $N+1$ for the second order. Formally, we write

$$G_{xx}^N = \frac{\lambda^{N-1}}{W}\mathbf{T}_N + \frac{\lambda^{N+1}}{W}(\mathbf{T}_{N+2}^a + \mathbf{T}_{N+2}^b) \tag{17}$$

with

$$\mathbf{T}_N = \prod_{i=1}^{N} L_x^i \tag{18}$$

$$\mathbf{T}_{N+2}^a = \sum_{k=1}^{N-1} L_x^k L_x^{k+1} \prod_{i=1}^{N} L_x^i \tag{19}$$

$$\mathbf{T}_{N+2}^b = \sum_{k=1}^{N}\sum_{l=k}^{N}\left[\sum_{\Delta x=\pm 1}\left(\prod_{1}^{k} L_x^\alpha\right)\exp[2\pi\mathrm{i}(\alpha k+\vartheta)]\left(\prod_{k}^{l} L_{x+\Delta x}^\beta\right)\exp[-2\pi\mathrm{i}(\alpha l+\vartheta)]\right.$$
$$\left.\times\left(\prod_{l}^{N} L_x^\gamma\right) + \sum_{NN\,y}{}' \prod_{1}^{k} L_x^\alpha \prod_{k}^{l} L_y^\beta \prod_{l}^{N} L_x^\gamma\right]. \tag{20}$$

(Both the terms of order $N+2$ were not given correctly in [8].) Obviously, the ϑ parameter does not influence the localisation length as long as the disorder and strip width are large enough to prevent hopping around the strip with a non-vanishing winding number.

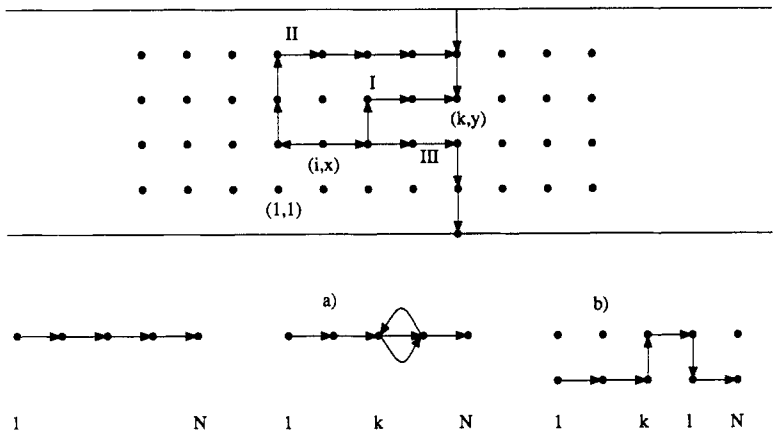

Figure 3. Three paths contained in the locator series for the resolvant operator $G_{ix,ky}$ (top) and the ones of leading order actually used in our expansion (17) (bottom).

The next step is to perform the configurational average $\prod_{ix}\int_{-1/2}^{1/2} \mathrm{d}\tau_x^i(\ldots)$. Unfortunately this is technically impossible without formally expanding the logarithm in (14) in powers of $\lambda = W^{-1}$. This procedure works for T_N and T_{N+2}^a:

$$\lim_{N\to\infty}\frac{1}{N}\Re\langle\ln\lambda^N\mathsf{T}_N\rangle = \ln\lambda + 1 + \frac{E}{W}\ln\frac{\frac{1}{2}+E/W}{\frac{1}{2}-E/W} - \frac{1}{2}\ln\left[\frac{1}{4}-\left(\frac{E}{W}\right)^2\right] \tag{21}$$

$$\lim_{N\to\infty}\frac{\lambda^2}{N}\Re\langle\frac{\mathsf{T}_{N+2}^a}{\mathsf{T}_N}\rangle = \lambda^2\left[\left(\ln\frac{\frac{1}{2}+E/W}{\frac{1}{2}-E/W}\right)^2 - \pi^2\right]$$

but we have to impose the condition $|\,E/W\,| < 0.097\ldots$ in order to keep the series

$$\sum_{k=1}^{N}\sum_{l=k}^{N}K^{l-k-1} = K^{-1}\left(\frac{N}{1-K} + \frac{1-K^N}{(1-K)(1-K^{-1})}\right) \qquad K = \langle L\rangle\langle L^{-1}\rangle \tag{22}$$

emerging during the evaluation of T_{N+2}^b convergent. Keeping this in mind, we may write (20) as

$$\lim_{N\to\infty}\frac{1}{N}\lambda^2\left\langle\frac{\mathsf{T}_{N+2}}{\mathsf{T}_N}\right\rangle = \lambda^2\langle L\rangle^2K^{-1}$$
$$\times\left((\kappa-4)\frac{1}{1-K} + \frac{1}{1-\exp[-2\pi i\alpha]K} + \frac{1}{1-\exp[2\pi i\alpha]K}\right). \tag{23}$$

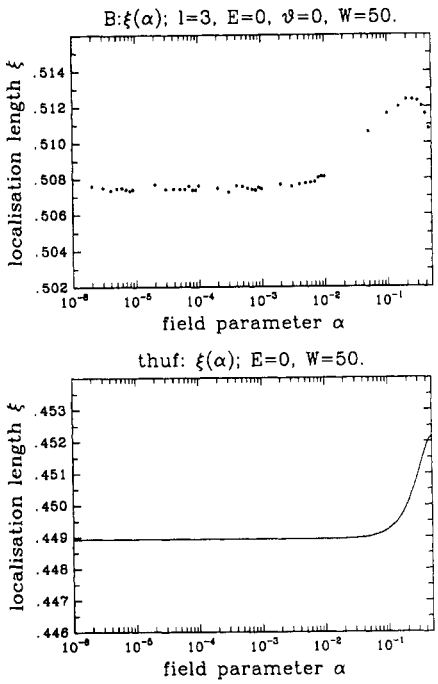

Figure 4. $\xi(\alpha)$ for $W = 50$ numerically (top) and as calculated perturbatively (bottom). Deviations occur mainly in regions of α, where the numerical method becomes unreliable ($\alpha < 10^{-5}$ or $\alpha = O(1)$).

Finally, we arrive at the following expression for $\xi_l(E, W, B)$:

$$\begin{aligned} -\gamma_1(E,W,B) = &-\ln W \\ &+1+\frac{E}{W}\ln\frac{\frac{1}{2}+E/W}{\frac{1}{2}-E/W}-\tfrac{1}{2}\ln\left[\frac{1}{4}-\left(\frac{E}{W}\right)^2\right] \\ &+\lambda^2\left[\left(\ln\frac{\frac{1}{2}+E/W}{\frac{1}{2}-E/W}\right)^2-\pi^2\right] \\ &+\lambda^2\left((\kappa-4)M(E/W,0)+M(E/W,B)+M(E/W,-B)\right) \\ &+O(\lambda^4) \end{aligned} \tag{24}$$

where we used the abbreviations

$$\begin{aligned} M\left(\frac{E}{W},B\right) &= \Re\left\{\langle L\rangle^2\frac{1}{K(1-\exp[2\pi i\alpha]K)}\right\} \\ &= \frac{W}{E}\frac{\ln x[1-(E/W)A(x,\alpha)]-\pi(E/W)(\pi\cos 2\pi\alpha+\ln x\sin 2\pi\alpha)}{[1-(E/W)A(x,\alpha)]^2+[(\pi\cos 2\pi\alpha+\ln x\sin 2\pi\alpha)(E/W)]^2} \end{aligned} \tag{25}$$

and

$$x=\frac{\frac{1}{2}+E/W}{\frac{1}{2}-E/W} \qquad A(x,\alpha)=\ln x\cos 2\pi\alpha-\pi\sin 2\pi\alpha.$$

As can be seen from figure 2, the result reproduces the numerics for large W. But as figure 4 tells us we did not reach our ultimate goal: the perturbative localisation length describes the field dependence in the numerically reliable region only qualitatively; the perturbative error is much larger than the variation of ξ with B.

As our problem seems to be well out of reach of perturbation theory, we now turn to the Lloyd model where some 'exact' methods apply.

4. The Lloyd model

The Lloyd model, characterised by a Hamiltonian of type (1) and Lorentzian disorder

$$p(\varepsilon_x^i) = \frac{\Gamma}{\pi}\frac{1}{(\varepsilon_x^i)^2 + \Gamma^2} \tag{26}$$

was invented by P Lloyd [11] in 1969 who was able to calculate exactly the averaged single-particle Green function and thereby the density of states. Based on the previous work of Herbert and Jones [4], Thouless [16] succeeded in deriving a formula for ξ_l for strictly 1D systems, the so-called HJT formula,

$$\cosh\gamma_1 = \frac{1}{4J}\left\{\left[(2J+E)^2+\Gamma^2\right]^{1/2} + \left[(2J-E)^2+\Gamma^2\right]^{1/2}\right\} \tag{27}$$

Johnston and Kunz [8] tried to generalise (27) for quasi-1D systems arriving at

$$\cosh\gamma_1 = \min_j \frac{1}{4J}\left\{\left[(2J+(E-J\alpha_j))^2+\Gamma^2\right]^{1/2} + \left[(2J-(E-J\alpha_j))^2+\Gamma^2\right]^{1/2}\right\} \tag{28}$$

where the minimum is to be taken over all eigenvalues α_j of the discretised Laplacian in d dimensions. As these are bounded by $|\alpha_j| < 2(d-1)$, (28) predicts complete localisation for arbitrary small disorder and for any spatial dimension which is not only non-physical, but also in contradiction with later numerical work by Bulka *et al* [14] and the author [15].

Although Thouless [16] observed a formal error in an argument about analytic continuation by Johnston and Kunz [6] designed to avoid the replica trick, this error is not responsible for the discrepancies between (28) and the numerical results. For a complete discussion, see [15].

Another weak point in the derivation of (28) seems to be the expansion (16) which enters into the calculation up to infinite order. As we will see later, such an objection is unjustified.

In any case (28) turns out not to be as 'exact' as it is supposed to be. The reason for this is not yet understood.

To investigate the disagreement between (28) and the numerical results, we have to re-examine the analytics, as cutting the Lorentz distribution at R for programming purposes only leads to corrections of the order R^{-2} to ξ_l, but does not produce any quasi-extended states on its own. Since the numerical analysis reveals the delocalising action of the magnetic field, we include it in our calculation right from the beginning.

Again, we start from formula (14) for the localisation length. The averaging procedure, (cf [8]) yields

$$\langle \ln G_{xy}^N(z)\rangle = \ln {}^0G_{xy}^N(z+\mathrm{i}\Gamma). \tag{29}$$

where the upper left index ^{0} means $\varepsilon_x^i \to \overline{\varepsilon_x^i} = 0$. Equation (29) already relies on what Johnston and Kunz [8] call an *assumption* about analytic continuation which, however, can be shown to hold at least for finite strip lengths N (cf [15]).

In order to calculate ${}^0G_{xy}^N(z)$ we make use of the identity

$$G^N = \left[A_{N+1}^{\dagger}(E+\mathrm{i}\varepsilon)\right]^{-1} \tag{30}$$

which follows by a straightforward calculation from the *ansatz*

$$\mathbf{T}(N) = \begin{pmatrix} A_{N+1} & B_{N+1} \\ A_N & B_N \end{pmatrix}. \tag{31}$$

(cf [8] or [15]). ${}^0A_{N+1}$ is easy to calculate for $B = 0$ (by merely diagonalising the ${}^0\mathbf{T}_i$), as in this case ${}^0\mathbf{T}_i = {}^0\mathbf{T}_j$ for all i, j. For a general α, the product ${}^0\mathbf{T}(N)$ is difficult to compute, but for rational fields ($\alpha = p/q$) we have

$$\prod_{i=1}^{q} {}^0\mathbf{T}_i = \prod_{i=q+1}^{2q} {}^0\mathbf{T}_i = \ldots \equiv {}^0\,\mathbf{T}^q. \tag{32}$$

Thus, it is sufficient to diagonalise the ${}^0\mathbf{T}^q$ to get $({}^0\mathbf{T}(N))_{11}$. Writing ${}^0\mathbf{T}(N)$ as

$${}^0\mathbf{T}(N) = S \begin{pmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{pmatrix}^{N/q} S^{-1} \tag{33}$$

we get for the eigen-matrix λ^+ (which is the only one we need, as transfer matrices are symplectic)

$$\lambda^+ = \exp(\cosh^{-1}(Z_q(\alpha))) \tag{34}$$

where the eigenvalues of $Z_q(\alpha)$ are recursively given by

$$\begin{aligned} \alpha_{k,j} &= a_j^{(k-1)}(z)\alpha_{k-1,j} - \alpha_{k-2,j} \\ \beta_{k,j} &= a_j^{(k-1)}(z)\beta_{k-1,j} - \beta_{k-2,j} \Rightarrow Z_{q,j}(\alpha) = \tfrac{1}{2}(\alpha_{q,j} + \beta_{q,j-1}) \end{aligned} \tag{35}$$

with starting values

$$\alpha_0 = 1 \qquad \alpha_1 = a^{(1)} \qquad \beta_0 = 0 \qquad \beta_1 = -1 \tag{36}$$

and the

$$a_j^{(k)}(z) = 2\cos 2\pi(\alpha k - j/l) + E + \mathrm{i}\varepsilon \tag{37}$$

being the eigenvalues of ${}^0P_k(\alpha)$ (cf (12)).

Knowing how to calculate the $\lambda^{\pm}$ we can now write

$$\begin{aligned} \Re\frac{1}{N}\langle \ln G_{xy}^N(z)\rangle = \frac{1}{N}\Re\Big\{\Big\langle \ln \sum_k U_{xk}\big[S_{11,k}(\lambda_k^+)^{N/q}(S^{-1})_{11,k} \\ + S_{12,k}(\lambda_k^-)^{N/q}(S^{-1})_{21,k}\big]^{-1}U_{yk}^*\Big\rangle\Big\} \end{aligned} \tag{38}$$

$$\begin{aligned} &\xrightarrow{N\to\infty} -\min_j \frac{1}{q}\Re\left\{\ln \lambda_j^+(q)\right\} \\ &= -\gamma_1(E, W, B) \end{aligned}$$

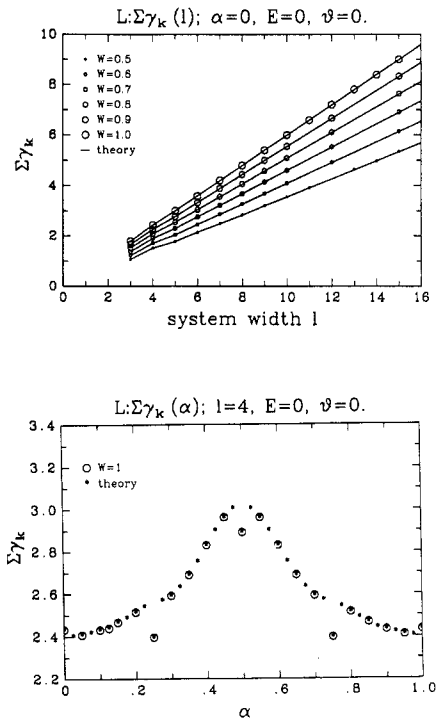

Figure 5. Comparison of simulation and theory for the sum of all Lyapunov exponents, $\sum \gamma_k$, as a function of the system width l (top) and α (bottom). (The agreement is that good because the numerical calculations were carried out to obtain γ_1 alone—of course, $\sum \gamma_k$ converges much faster.)

(U being a unitary matrix diagonalising all ${}^0P_k(\alpha)$ simultaneously) with

$$\min_j \frac{1}{q}\Re\{\ln \lambda_j^+(q)\} = \min_j \frac{1}{q}\Re\{\cosh^{-1} Z_{q,j}(\alpha)\} \tag{39}$$

$$= \min_j \frac{1}{q} \cosh^{-1} \tfrac{1}{2}\left\{\left[(x_j+1)^2 + y_j^2\right]^{1/2} + \left[(x_j-1)^2 + y_j^2\right]^{1/2}\right\}$$

where the x_j and y_j denote the real and imaginary parts of $Z_{q,j}(\alpha)$, respectively.

To check this, we first plot the sum of all Lyapunov exponents as a function of system width l and magnetic field α (figure 5) and compare it with the numerical values. The agreement is as good as we would expect it from an 'exact' solution.

Secondly, we look at the result for γ_1 alone. As the solution (39) reduces to (28) for $\alpha = 0$, there is still the prediction of complete localisation for all $\Gamma \neq 0$, which remains valid for $\alpha \neq 0$. However, comparing our numerical results for $\xi(\alpha)$ for the box distribution with the curve resulting from (39), we observe qualitative agreement (see figure 6). They could not necessarily be expected to coincide as they belong to different probability distributions. The exact correspondence of the extrema seems to indicate that our method correctly describes the field dependence of ξ. Unfortunately, we cannot peek into the interesting region of $\lg \alpha < -5$ as treating qs much greater than 200 requires enormous computer effort.

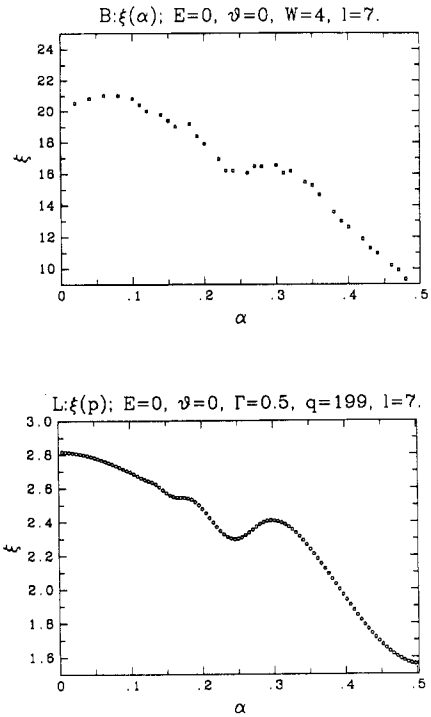

Figure 6. A comparison of $\xi(\alpha)$ as computed numerically for the box distribution (top) and analytically for the Lorentz distribution (bottom), where $\alpha = p/q$, reveals qualitative agreement between the two.

5. Conclusions

After all, it is possible to calculate perturbatively the localisation length in the limit of large disorder. However, the functional dependence of the localisation length on the field parameter α can be calculated only qualitatively from this expansion, since only the second and higher non-vanishing orders are actually field-dependent. So the calculation leading to figure 4 is to be viewed as a first-order approximation.

An 'exact' solution for the Lloyd model (which has been believed to be incorrect for five years) is able to correctly describe the field dependence of ξ and will perhaps be able to shed some light on the transition from the orthogonal to the unitary case; but this question will have to be the subject of further investigation.

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Note added in proof. When the recursion relation solving the Lloyd model (essentially equations (35) and (39)) is used to calculate the energy spectra of the localisation length in the *low* disorder limit, the entire magnetic sub-band structure is reproduced in its disorder-broadened version which was predicted in 1976 by Hofstadter [17] for rational values of the field parameter $\alpha = p/q$. Figure A.1 shows how these sub-bands develop for $q = 1, \ldots, 8$. This is another independent check for the

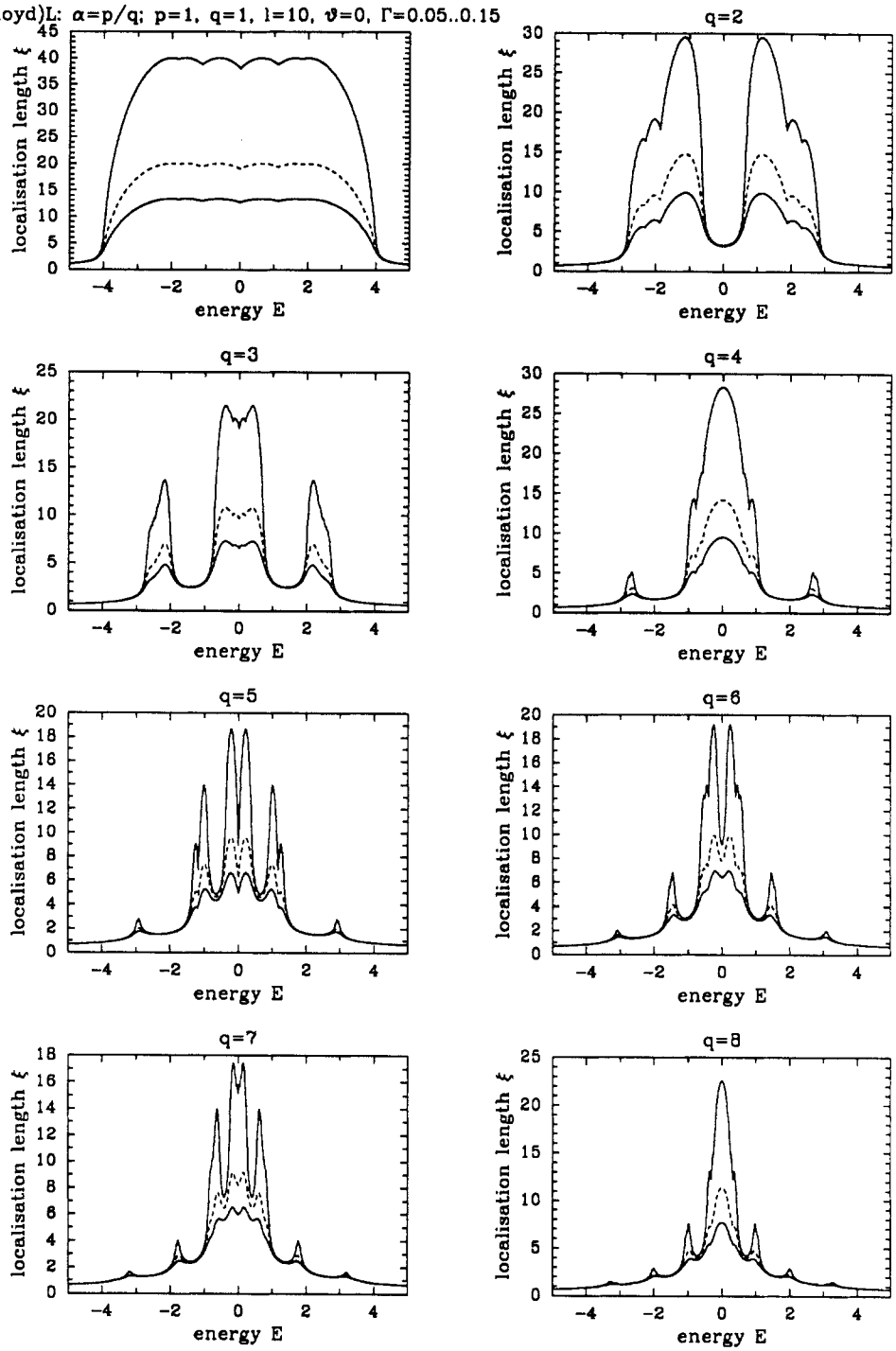

Figure A.1. The localisation length ξ is plotted as a function of the energy for eight values of the field parameter $\alpha = p/q$, $p = 1$, $q = 1, \ldots, 8$ with the disorder Γ as a parameter ($\Gamma = 0.05, 0.10, 0.15$) as calculated from the solution of the Lloyd model (equations (35)–(39)). The spectra consist of q disorder-broadened magnetic sub-bands as predicted by Hofstadter [17].

reliability of (39) which now seems to be able to generate the correct functions $\xi(x)$ for all system parameters x except the width l.

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